SOLUTION TO THE PROBLEM OF TRANSIENT THERMAL CONTACT BETWEEN TWO DIFFERENT DISKS

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In performing the Laplace transformation, the problem of representation by changing from a series expansion in one system of eigenfunctions to a series expansion in another system is reduced to an infinite system of linear equations. The Kramer rule is used for constructing the original function. This method is particularly suitable for the analysis of a regular heating mode.

In practical thermotechnical design one encounters problems involving systems with a sudden inhomogeneity in the material in the direction normal to the flow of heat [1-3]. The spatial inhomogeneity in these problems is due mainly not to the nature of the boundary conditions but to the structure of the region for which the solution is sought. In these cases the heat propagates deeper into a structure, essentially along components whose thermal conductance is high. In a rigorous solution of such problems, however, it is necessary to take into account also the heat storing capacity of contiguous components whose thermal conductance is low.

We will analyze this problem on a simple mathematical model of such systems:

$$\frac{\partial t_i}{\partial \tau} = a_i L_i t_i + u_i; \quad [z_i \in (0,\ \delta_i), \quad r \in (r_0,\ (r_0+1))], \tag{1}$$

$$L_i = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z_i^2} + \frac{\partial}{r\partial r}\right), \quad u_i = u_i(\tau,\ r,\ z_i), \quad i = 1,\ 2,$$

$$\left.\frac{\partial t_i}{\partial r}\right|_{r=r_\Delta} = (-1)^\Delta h_{i\Delta}[t_i|_{r=r_\Delta} - t_{\Delta i}], \quad r_\Delta = (r_0+\Delta), \quad \Delta = 0,\ 1, \tag{2}$$

$$\left.\frac{\partial t_i}{\partial z_i}\right|_{z_i=\delta_i} = g_i[t_{\delta i} - t_i|_{z_i=\delta_i}], \tag{3}$$

$$t_1|_{z_1=0} = t_2|_{z_2=0} - f\left.\frac{\partial t_2}{\partial z_2}\right|_{z_2=0}, \tag{4}$$

$$-\left(\left.\frac{\partial t_1}{\partial z_1}\right|_{z_1=0} + k\left.\frac{\partial t_2}{\partial z_2}\right|_{z_2=0}\right) = u = u(r,\ \tau), \tag{5}$$

$$t_i|_{\tau=0} = F_i = F_i(r,\ z_i). \tag{6}$$

We note that $t_{\Delta i}$ and $t_{\delta i}$ in (2) and (3) can be functions of z_i, τ and of r, τ, respectively, while (4) and (5) represent the conditions of nonideal contact (when $f \neq 0$) with heat emission at the surfaces.

The special case of system (1)-(6) with an ideal contact and equal Biot numbers has been considered in [4] (a two-layer rectangular beam with boundary conditions of the first kind) and by Kirshbaum in [3] (contact between two semiinfinite cylinders).

An essential feature of our problem is that $h_{1\Delta}$ and $h_{2\Delta}$ can be different, which creates major difficulties in treating conditions (4) and (5) when the separation-of-variables method is used.

We will assume that functions u, u_i, and F_i satisfy the Dirichlet conditions and are expressible in terms of series of the kind

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$$y_i = \Sigma_n y_{in} R_{in}, \quad \Sigma_n a_n = \sum_{n=1}^{\infty} a_n \tag{7}$$

in eigenfunctions of the following eigenvalue problem:

$$\frac{\partial^2 R_{in}}{\partial r^2} + \frac{\partial R_{in}}{r \partial r} + \alpha_{in}^2 R_{in} = 0, \tag{8}$$

$$\left.\frac{\partial R_{in}}{\partial r}\right|_{r=r_\Delta} - (-1)^\Delta h_{i\Delta} R_{in}\Big|_{r=r_\Delta} = 0. \tag{9}$$

Let also v_i be an arbitrary function which satisfies (2)-(3) and is expressible as series of the (7) kind, together with its Laplacian $L_i v_i$.

In this case, performing the Laplace transformation with respect to τ (with the parameter p defined as in [5]) and subsequently separating the variables in order to find the transforms $\bar{t}_i$, one obtains

$$\bar{t}_i = \bar{v}_i + \vartheta_i; \quad \vartheta_i = \Sigma_n R_{in}(C_{in} Z_{in} + \Phi_{in}), \tag{10}$$

where

$$Z_{in} = (\operatorname{ch} Y_{in} + g_i q_{in}^{-1} \operatorname{sh} Y_{in}), \quad q_{in} = \sqrt{p a_i^{-1} + \alpha_{in}^2}, \tag{11}$$

$$\Phi_{in} = \varphi_{in} - \frac{g_i \varphi_{in\delta} + \varphi'_{in\delta}}{(g_i + q_{in}) \exp Y_{in}}, \quad Y_{in} = q_{in}(\delta_i - z_i) \tag{12}$$

and φ_{in} is the particular solution to the equation

$$\frac{d^2 \varphi_{in}}{dz_i^2} - q_{in}^2 \varphi_{in} = U_{in}; \quad U_i = \frac{p\bar{v}_i - \bar{u}_i - F_i}{a_i} - L_i \bar{v}_i. \tag{13}$$

In (12) and further on we use the following kind of notation:

$$\theta_{in}|_{z_i=b_i} = \theta_{inb}, \quad \left.\frac{d\theta_{in}}{dz_i}\right|_{z_i=b_i} = \theta'_{inb}; \quad \left.\frac{d\varphi_{in}}{dz_i}\right|_{z_i=\delta_i} = \varphi'_{in\delta}.$$

For determining the constants C_{in} in (10), one must consider the "contiguity" conditions at $z_i = 0$, which follow from (4) and (5):

$$\vartheta_1 - \vartheta_2 + f\frac{\partial \vartheta_2}{\partial z_2} = \bar{v}_2 - \bar{v}_1 - f\frac{\partial \bar{v}_2}{\partial z_2} = V = \Sigma_n V_n R_{2n}, \tag{14}$$

$$-\left(\frac{\partial \vartheta_1}{\partial z_1} + k\frac{\partial \vartheta_2}{\partial z_2}\right) = \bar{u} + \frac{\partial \bar{v}_1}{\partial z_1} + k\frac{\partial \bar{v}_2}{\partial z_2} = U = \Sigma_n U_n R_{1n}. \tag{15}$$

It is convenient here to utilize the possibility of series expansion

$$R_{in} = \Sigma_m b_{inm} R_{jm} \quad \text{for} \quad i \neq j, \quad r_0 < r < (r_0 + 1). \tag{16}$$

Specifically, for $r_0 = 0$ the formula in [6] yields

$$R_{in} = J_{inr} \text{ and } b_{inm} = \frac{2(h_{21} - h_{11})\alpha_{jm}^2 J_{in1}}{(\alpha_{jm}^2 - \alpha_{in}^2)(\alpha_{jm}^2 + h_{j1}^2) J_{jm1}},$$

where $J_{inr} = J_0(\alpha_{in} r)$ is the zeroth-order Bessel function of the first kind and α_{in} are the roots of the equation

$$\left.\frac{dJ_{inr}}{dr}\right|_{r=1} + h_{i1} J_{in1} = 0.$$

When $r_0 \to \infty$, however, one obtains

$$R_{in} = \cos \alpha_{in}\rho + h_{i0}\alpha_{in}^{-1} \sin \alpha_{in}\rho, \quad 0 < \rho = (r - r_0) < 1,$$

where α_{in} are the roots (all real) of the equations

$$\alpha_{in} \operatorname{tg} \alpha_{in} = P_{in}^{-1}, \quad P_{in} = (1 - h_{i0} h_{i1} \alpha_{in}^{-2})(h_{i1} + h_{i0})^{-1}.$$

Moreover, for u = in and v = jm we have

$$b_{um} = 2\alpha_v(F_{vu} - F_{uv}) S_u [S_v G_v \alpha_u(\alpha_u^{-2} - \alpha_v^{-2})]^{-1},$$

where

$$G_v = [2h_{j0} + P_v(\alpha_v^2 - h_{j0}^2) + (\alpha_v^2 + h_{j0}^2)(1 + \alpha_v^2 P_v^2)],$$
$$F_{vu} = \alpha_u^{-2} h_{i0} + P_u(1 + h_{i0} h_{j0} \alpha_v^{-2} + P_v h_{j0}),\ S_u = \sin \alpha_u.$$

On the other hand, it follows from (15) and (16) that

$$C_{1n} = Q_{1n}^{-1}[\psi_n - k\Sigma_m C_{2m} Q_{2m} b_{2mn}],\ Q_{in} = q_{in}\,\mathrm{sh}\, Y_{in0} + g_i\,\mathrm{ch}\, Y_{in0},\ \psi_n = U_n + \Phi'_{1n0} + k\Sigma_m \Phi'_{2m0} b_{2mn}. \quad (17)$$

Furthermore, by virtue of (17) and (16), it is possible to derive from (14) an infinite system of equations which are linear with respect to $B_n = C_{2n} Q_{2n} R_{2n1}$:

$$B_n + \Sigma_l^{(n)} B_l \Sigma_m T_{m,ln} = \Psi_n = A_n + P_n,$$

where

$$\Sigma_l^{(n)} a_l = \Sigma_l a_l - a_n,$$
$$T_{m,ln} = \frac{R_{2n1} Z_{1m0} k}{R_{2l1} Q_{1m} H_n} b_{2lm} b_{1mn};\quad H_n = \frac{Z_{2n0}}{Q_{2n}} + f + \Sigma_m T_{m,nn} H_n, \quad (18)$$
$$\Psi_n = H_n^{-1} R_{2n1}[V_n - \Phi_{2n0} + f\Phi'_{2n0} + \Sigma_m b_{1mn}(\Phi_{1m0} + \psi_m Z_{1m0} Q_{1m}^{-1})],$$

and the components of Ψ_n are one (P_n) a function of and one (A_n) independent of $\bar{v}_i$, $\bar{u}_i$, and $\bar{u}$; here $R_{2n1} = R_{2n}|_{r=r_1}$.

With p assumed real, as will be proved later on, the method in [8] will yield the estimates

$$\Sigma_n \Sigma_l^{(n)} \sigma_{ln}^2 < \infty,\quad \Sigma_n \Sigma_l^{(n)} \sigma_{ln} < \infty, \quad (19)$$

where

$$\sigma_{ln} = \Sigma_m T_{m,ln},$$

and for the first of which, pertaining to an infinite system, all theorems applicable to a finite number of unknowns [9] will retain their validity. For instance, B_n can be determined according to the Kramer rules and

$$C_{2n} = (R_{2n1} D Q_{2n})^{-1} \Sigma_l (-1)^{n+l} D_{ln} \Psi_l,$$

where D is an infinite determinant whose n-th row is filled with the coefficients of B_l in the increasing order of l from the n-th equation of system (18), and where D_{ln} is obtained from D by removing the n-th column and the l-th row. However, both D and D_{ln} are normal determinants, by virtue of the obvious convergence of the infinite product of their diagonal elements (1-elements) and also by virtue of the second estimate in (19). As is well known, the same statements apply to normal as to finite determinants [9].

In order to find the t_i functions from their transforms, it remains now to apply the inversion theorem and the Cauchy residue theorem to the contour of integration which excludes the right-hand half-plane [5]. As it turns out,

$$t_i = F_i + \Sigma_l \lim_{p \to \mu_l} \left\{ \left(\frac{dD}{dp} \right)^{-1} \Sigma_n \Omega_n \Sigma_m (-1)^{n+m} D_{nm} \omega_{im} \right\}, \quad (20)$$

where

$$\Omega_n = \frac{P_n}{p} + A_n(e^{p\tau} - 1) + e^{p\tau} \int_0^{\tau} P_n e^{-p\tau}\, d\tau,$$
$$\omega_{2m} = R_{2m} Z_{2m} Q_{2m},\qquad \omega_{1m} = k\Sigma_{s1} b_{2ms} R_{1s} Z_{1s} Q_{1s}^{-1}.$$

In (20) μ_l are the roots of the transcendental equation

$$D(\mu) = D|_{p=\mu} = 0. \quad (21)$$

All roots μ_l in (20) are assumed to be simple ones, following an analysis of the Eq. (21) structure and on the basis of analogy with conventional solutions in heat conduction theory. An analysis of very specific cases with multiple roots of Eq. (21) does not present any fundamental difficulties.

Since all coefficients in Eq. (21) are real, hence all its complex roots must form conjugate pairs. If one extends the method shown in [5], however, then it can be proved that Eq. (21) has no complex roots.

Indeed, let Γ be a root of Eq. (21), i.e., $D(\Gamma) = 0$.

Then for $p = \Gamma$ there exists a nonzero solution to system (17)-(18) such as C_{il}, for example, and

$$\Theta_i = \Sigma_n Z_{in} R_{in} C_{in}$$

are obviously the solutions to the homogeneous equations

$$a_i L_i \Theta_i = \Gamma \Theta_i \tag{22}$$

with homogeneous boundary conditions according to (2)-(5).

Let now Γ and γ be two different roots of Eq. (21) and let functions θ_i correspond to the functions Θ_i when Γ is replaced by γ. Then Eqs. (22) and the analogous equations for θ_i yield

$$(\Gamma-\gamma)\sum_{i=1,2} k^{(i-1)} a_i^{-1} \int_0^{\delta_i} dz_i \int_{r_0}^{(r_0+1)} r\Theta_i\theta_i dr = B = \sum_{i=1,2} k^{(i-1)} \int_0^{\delta_i} dz_i \int_{r_0}^{(r_0+1)} rdr\,[\theta_i L_i \Theta_i - \Theta_i L_i \theta_i]. \tag{23}$$

Integrating the right-hand side of (23) and using the boundary conditions, we obtain

$$B = \sum_{i=1,2} k^{(i-1)} \int_{r_0}^{(r_0+1)} rdr \left[\Theta_i \frac{\partial\theta_i}{\partial z_i} - \theta_i \frac{\partial\Theta_i}{\partial z_i}\right]_{z_i=0} = 0.$$

It follows from here that Γ and γ cannot be complex-conjugate quantities. If they were, then θ_i and Θ_i would have been conjugate too and the left-hand side of (23) would be positive in violation of the last equality ($B = 0$).

Therefore, Eq. (21) can have only positive roots and this justifies, specifically, the assumption on which estimates (19) have been based. The roots can be found conveniently by the Newton method.

As is evident from (18), (21) and (20) are particularly useful for the analysis of a regular heating mode [7] in a system where

$$|T_{1,ln}| \gg |\Sigma_m^{(1)} T_{m,ln}|, \quad l,\ n = 1,\ 2,\ \ldots,\ \infty,$$

which is possible when

$$g_1 \ll 1, \quad |\alpha_{11}^2 + \mu_1| \ll 1, \quad \delta_1 |\alpha_{11}^2 + \mu_1| \ll 1.$$

The last inequalities are valid if

$$h_{11} \ll 1, \quad g_1 \ll 1, \quad \delta_i \ll 1, \tag{24}$$

i.e., at a contact between two different disks at moderate rates of heat transfer to the ambient medium.

When (24) is satisfied, then (21) and (18) yield

$$D \approx 1 + \frac{Z_{110}}{Q_{11}} k \sum_n b_{2n1} b_{11n} \left(\frac{Z_{2n0}}{Q_{2n}} + f\right)^{-1} = 0. \tag{25}$$

from which the roots μ_l can be determined.

In (20), moreover,

$$D_{nn} = -\left(\frac{Z_{2n0}}{Q_{2n}} + f\right)^{-1}; \qquad D_{nl} = \frac{(-1)^{n+l} Q_{11} D_{nn} D_{ll}}{Z_{110} k b_{11n} b_{2l1}}, \quad n \neq l.$$

For faster calculations, the problem can be programmed for a digital computer. Evidently, programming the analytical formulas derived here is preferable to direct numerical integration of the system

(1)-(6), especially in the case of large cylinders and in the case of a long process.

NOTATION

T_i	is the temperature of the i-th cylinder;
$\beta_{i\Delta}$, γ_i	are the coefficients of heat transfer to the medium;
γ^{-1}	is the thermal contact resistance between cylinders;
$\bar{k}_i$	is the thermal conductivity;
$\bar{a}_i$	is the thermal diffusivity;
R, r_0R	are the width and the inside radius of a ring;
d_i	is the height of a cylinder;
$\bar{z}_i$, $\bar{r}$	are the space coordinates;
$\bar{\tau}$	is the time;
$(T_M - T_0)$	is the maximum difference between ambient temperature and initial temperature of cylinders;
$t_i = (T_i - T_0)/(T_M - T_0)$, $z_i = \lvert\bar{z}_i\rvert/R$, $r = \bar{r}/R$, $\delta_i = d_i/R$, $\tau = \bar{a}_1\bar{\tau}/R^2$, $h_{i\Delta} = \beta_{i\Delta}R/\bar{k}_i$, $g_i = \gamma_i R/\bar{k}_i$, $k = \bar{k}_2/\bar{k}_1$, $a_i = (\bar{a}_2/\bar{a}_1)^{i-1}$, $f = \bar{k}_1/\gamma R$	are the dimensionless variables and characteristics;
$t_{\delta i}$, $t_{\Delta i}$	is the ambient temperature at $z_i = \delta_i$ and at $r = r_\Delta$, respectively.

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